Motion in Space

Let $\mathbf{r}(t) = \langle f(t), h(t), g(t)\rangle$. Think of $\mathbf{r}(t)$ the position of a spacecraft at time t.

$\mathbf{r}'(t) = \langle f'(t), h'(t), g'(t)\rangle = \mathbf{v}(t)$ is the *velocity* of $\mathbf{r}(t)$.

The *arc length of* $\mathbf{r}(t)$ *from time* a *to* t is $\int_a^t |\mathbf{r}'(u)|du$.

Hence *speed* of the spacecraft is $v(t) = \frac{d}{dt}\int_a^t |\mathbf{r}'(u)|du = |\mathbf{r}'(t)|$ which also is the magnitude of the velocity, $\mathbf{v}(t)$.

The *acceleration* of $\mathbf{r}(t)$ is $\mathbf{r}''(t) = \langle f''(t), h''(t), g''(t)\rangle = \mathbf{v}'(t) = \mathbf{a}(t)$. *Force* is $m\mathbf{a}$.

The *unit tangent* of $\mathbf{r}(t)$ is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

Since the unit tangent has length 1, $\mathbf{T}'(t)$ and $\mathbf{T}(t)$ are orthogonal. The *unit normal* is $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.

The *unit binormal* $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

Unless forced one never wants to find $\mathbf{N}(t)$ and/or $\mathbf{B}(t)$ as a function of time. (Differentiating $\mathbf{T}(t)$ can be painful.) One should determine $\mathbf{T}(a)$ and $\mathbf{N}(a)$ and then take the cross product. But unless you are forced to find $\mathbf{N}(a)$ there is an easier way to find $\mathbf{B}(t)$: I.e., $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{\mathbf{r}'\times\mathbf{r}''}{|\mathbf{r}'\times\mathbf{r}''|}$. In turn we can use $\mathbf{B}$ and $\mathbf{T}$ to find $\mathbf{N}$; $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

The *tangent line* at $t = t_0$ is the line though $\mathbf{r}(t_0)$ in the direction of $\mathbf{r}'(t_0) = \mathbf{v}(t_0)$ or $\mathbf{T}(t_0)$. So $\mathbf{l}(t) - \mathbf{r}(t_0) = t\mathbf{r}'(t_0)$.

The *normal plane* at $t = t_0$ is the plane though $\mathbf{r}(t_0)$ with a normal in the direction of $\mathbf{r}'(t_0) = \mathbf{v}(t_0)$ or $\mathbf{T}(t_0)$. So $[\langle x, y, z\rangle - \mathbf{r}(t_0)] \cdot \mathbf{r}'(t_0) = 0$.

The *osculating plane* at $t = t_0$ is the plane though $\mathbf{r}(t_0)$ containing $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$. Hence a normal for this plane is $\mathbf{B}(t_0)$. So $[\langle x, y, z\rangle - \mathbf{r}(t_0)] \cdot \mathbf{B}(t_0) = 0$.

The *curvature* of a curve $\mathbf{r}(t)$ is $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{T}'(t)|}{v(t)} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. The curvature is always positive. The tighter the curve the larger the curvature.

We know that $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. Hence $\mathbf{a} = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) = v'(t)\mathbf{T}(t) + v(t)|T'(t)|\mathbf{N}(t)$. So $\mathbf{a}(t_0)$ breaks into two components; one in the direction of the unit tangent and one in the direction of the unit normal, $\mathbf{a}(t_0) = a_T\mathbf{T}(t_0) + a_N\mathbf{N}(t_0)$. The *tangential component* of acceleration (at time t_0) is a_T. The *normal component* of acceleration (at time t_0) is a_N. $\mathbf{a}(t_0)$ is orthogonal to $\mathbf{B}(t_0)$.

Note $\mathbf{a}(t) \cdot \mathbf{v}(t) = (a_T\mathbf{T}(t) + a_N\mathbf{N}(t)) \cdot v(t)\mathbf{T}(t) = a_T v$. So $a_T = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v(t)}$.

The normal component a_N is $v(t)|T'(t)|$. Since both these terms are always ‘positive, a_N is always positive. Now $|\mathbf{a}(t)|^2 = a_T^2 + a_N^2$, since $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthonormal. Hence $a_N = \sqrt{|\mathbf{a}(t)|^2 - a_T^2}$. But also $|T'(t)| = \kappa(t)v(t)$. So $a_N = \kappa v^2$.

Hence the plane determined by $\mathbf{T}$ and $\mathbf{N}$ and the plane determined by $\mathbf{v}$ and $\mathbf{a}$ are the same plane, the osculating plane. So the osculating plane is normal to $\mathbf{v} \times \mathbf{a}$ and $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$.